Appello 21/02/2024

mercoledì 6 marzo 2024 15:16

Es 2

X e Y due vv. aa. indipendenti identicamente distribuite di densità di probabilità

$$f(x) = C x^2 \mathbb{1}_{[0,1]}(x)$$

Calcolare:

1) C e la funzione di ripartizione congiunta delle componenti del vettore aleatorio (X,Y)

2) La densità di probabilità di $Z = \sqrt{X+Y}$

3) La matrice

$$\begin{pmatrix} Cov(X,X) & Cov(X,Y) & Cov(X,Z) \\ Cov(Y,X) & Cov(Y,Y) & Cov(Y,Z) \\ Cov(Z,X) & Cov(Z,Y) & Cov(Z,Z) \end{pmatrix}$$

Sol

Poiché

$$1 = \int_0^1 C x^2 \, dx$$

$\Rightarrow$

$$1 = C \left. \frac{x^3}{3} \right|_0^1$$

$\Rightarrow$

$$1 = \frac{C}{3} \Rightarrow C = 3$$

Siccome X e Y sono indipendenti, si ha

$$F_{(X,Y)}(x,y) = F_X(x) \cdot F_Y(y)$$

e inoltre

$$F_Y(y) = F_X(y) \quad \text{perché} \quad X \stackrel{d}{=} Y$$

Cominciamo col calcolare $F_X(x)$:

Poiché vale che

$$\frac{\partial}{\partial x} F_X = f(x)$$

$$\Rightarrow F_X = \int_{-\infty}^{x} f(x)\, dx$$

$$\Rightarrow F_X = \int_{-\infty}^{x} 3x^2 = 3\frac{x^3}{3} = x^3$$

e dunque

$$F_X = x^3 \cdot 1_{(0,1]}(x) + 1_{(1,+\infty)}(x)$$

Allora

$$F_Y = y^3 \cdot 1_{(0,1]}^{(y)} + 1_{(1,+\infty}(y)$$

e dunque

e dunque

$$F_{(X,Y)}(x,y) = F_X(x)\cdot F_Y(y) = \left(x^3 \cdot 1_{(0,1]}(x) + 1_{(1,+\infty)}(x)\right)\cdot\left(y^3\cdot 1_{(0,1]}(y) + 1_{(1,+\infty)}(y)\right)$$

2) Sappiamo che $Z = \sqrt{X+Y}$

Per semplicità poniamo $W \overset{def}{=} X+Y$ in modo che poi possiamo lavorare con la quantità $Z = \sqrt{W}$

Dunque

$W = X+Y$

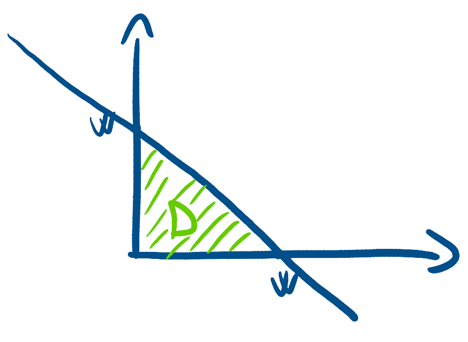
$\swarrow P(W \le w) = P(X+Y \le w)$

$$F_W(w) = \iint_D 3x^2 \cdot 3y^2 \, dx\, dy = \iint_D 9x^2y^2\, dx\, dy$$

Chi è D?

Sappiamo che

$$F_W(w) = P(W \le w) = P(X+Y \le w)$$
$$= P(Y \le w - x)$$



Per cui

$$\iint 9x^2y^2\, dx\, dy = 9\int^{w}\int^{w-x} x^2 y^2\, dy\, dx =$$

In cui

$$F_\psi(\psi) = \iint_D 9x^2 z^2 \, dx \, dz = 9 \int_0^\psi \int_0^{\psi - x} x^2 z^2 \, dz \, dx =$$

$$= 9 \int_0^\psi x^2 \, \frac{z^3}{3} \Big|_0^{\psi - x} dx =$$

$$= 9 \int_0^\psi x^2 \, \frac{(\psi - x)^3}{3} dx =$$

$$= 9 \int_0^\psi x^2 \, \frac{(\psi - x)^2 \cdot (\psi - x)}{3} dx =$$

$$= 3 \int_0^\psi x^2 (\psi^2 + x^2 - 2x\psi)(\psi - x) \, dx =$$

$$= 3 \int_0^\psi x^2 (\psi^3 - x\psi^2 + x^2\psi - x^3 - 2x\psi^2 + 2x^2\psi) \, dx =$$

$$= 3 \int_0^\psi x^2\psi^3 - x^3\psi^2 + x^4\psi - x^5 - 2x^3\psi^2 + 2x^4\psi \, dx =$$

$$= 3 \int_0^\psi x^2\psi^3 - 3x^3\psi^2 + 3x^4\psi - x^5 \, dx =$$

$$= 3 \left(\frac{x^3}{3}\psi^3 - 3\frac{x^4}{4}\psi^2 + 3\frac{x^5}{5}\psi - \frac{x^6}{6} \right) \Big|_0^\psi =$$

$$= 3 \left(\frac{\psi^6}{3} - \frac{3}{4}\psi^6 + \frac{3}{5}\psi^6 - \frac{\psi^6}{6} \right) =$$

$$= 3 \left(\frac{20\psi^6 - 45\psi^6 + 36\psi^6 - 10\psi^6}{60} \right) =$$

$$= 3\left(\frac{1}{60}w^6\right) = \frac{1}{20}w^6$$

Riassumendo

$$F_W(w) = \dots = \frac{1}{20}w^6$$

Da cui

$$f_W(w) = \frac{d}{dw}F_W = \frac{1}{20}\cdot 6 \cdot w^5 = \frac{3}{10}w^5$$

Allora

$$F_Z(z) = P(Z \leq z) = P(\sqrt{W} \leq z) = P(W \leq z^2) =$$

$$= \int_0^{z^2} f_W(w)\,dw = \int_0^{z^2} \frac{3}{10}w^5\,dw$$

$$= \frac{3}{10}\left.\frac{w^6}{6}\right|_0^{z^2} = \frac{3}{60}z^{12}$$

Da cui

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{3}{5}z^{11}$$

3) Siccome X e Y sono indipendenti

$\Rightarrow$ $Cov(X, Y) = 0$

$Cov(Y, X) = 0$

e poiché $X \overset{d}{=} Y$, si ha:

$$Cov(X,X) = Cov(Y,Y)$$

e

$$Cov(X,Z) = Cov(Z,X) = Cov(Z,Y) = Cov(Y,Z)$$

Dobbiamo calcolare allora:

- $Cov(X,X)$
- $Cov(Y,Y)$
- $Cov(X,Z)$
- $Cov(Y,Z)$
- $Cov(Z,X)$
- $Cov(Z,Y)$
- $Cov(Z,Z)$

Sapendo che $Cov(X,X) = Var(X)$, calcoliamo $Var(X)$:

$$Var(X) = E[X^2] - E^2[X]$$

Dunque

$$E[X] = \int_0^1 f_X(x) \cdot x \, dx = \int_0^1 3x^2 \cdot x \, dx = \int_0^1 3x^3 \, dx = \frac{3}{4}$$

$$E[X^2] = \int_0^1 f_X(x) \cdot x^2 \, dx = \int_0^1 3x^2 \cdot x^2 = \int_0^1 3x^4 = \frac{3}{5}$$

$$E[X^2] = \int_0^1 f_X(x) \cdot x^2 \, dx = \int_0^1 3x^2 \cdot x^2 = \int_0 3x^4 = \frac{3}{5}$$

$\Rightarrow$

$$Var(X) = \frac{3}{5} - \frac{9}{16} = \frac{48-45}{80} = \frac{3}{80}$$

Analogamente $Cov(Y,Y) = Var(Y)$:

$$Var(Y) = E[Y^2] - E^2[Y]$$

$\Rightarrow$

$$E[Y] = \int_0^1 f_Y(y) \cdot y \, dy = \int_0^1 3y^2 \cdot y = \int_0^1 3y^3 = \frac{3}{4}$$

$$E[Y^2] = \int_0^1 f_Y(y) \cdot y^2 \, dy = \int_0^1 3y^4 = \frac{3}{5}$$

$\Rightarrow$

$$Var(Y) = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}$$

Calcoliamo ora $Cov(X,Z)$:

$$Cov(X,Z) = E[XZ] - E[X] \cdot E[Z]$$

Dunque

$$E[X] = \int_0^1 f_X(x) \cdot x \, dx = \int_0^1 3x^3 = \frac{3}{4}$$

$$E[Z] = \int_0^1 f_Z(z) \cdot z \, dz = \int_0^1 \frac{3}{5} z^{11} \cdot z \, dz = \frac{3}{5} \cdot \frac{z^{13}}{3} \Big|_0^1 =$$

$$= \frac{1}{5}$$

$$E(XZ) = E\left[X\sqrt{X+Y}\right] =$$

$$= \int_0^1\int_0^1 3x^2 \cdot 3y^2 \cdot x \cdot \sqrt{x+y}\; dx\, dy =$$

$$= \int_0^1\int_0^1 9x^3 y^2 \sqrt{x+y}\; dx\, dy =$$

$$= 9\int_0^1\int_0^1 x^3 y^2 \sqrt{x+y}\; dy\, dx =$$

$$= \ldots = \frac{2}{27}$$

$\Rightarrow$

$$Cov(X,Z) = \frac{2}{27} - \frac{3}{4}\cdot\frac{1}{5} = \frac{2}{27} - \frac{3}{20}$$

Rimane $Cov(Z,Z)$:

$$Cov(Z,Z) = Var(Z)$$

$\Rightarrow$

$$Var(Z) = E[Z^2] - E^2[Z]$$

Però

$$E(Z^2) = E[X+Y] = E[X] + E[Y] = 2E(X)$$

$\Rightarrow$

$$Var(Z) = 2E[X] - E^2[Z] = 2\cdot\frac{3}{4} - \left(\frac{1}{5}\right)^2$$

∎

Es 4

$$F_\Theta(\theta) = c \int_{-\infty}^{\theta} \eta^2 (1-\eta) \cdot 1_{[0,1]}(\eta)\, d\eta$$

Dopo 4 osservazioni si osserva che il primo e secondo evento sono accaduti, mentre terzo e quarto non sono stati osservati. Calcolare la densità di probabilità e il valore che la massimizza

SOL

Risulta che la densità di probabilità a posteriori è data da:

$$\pi_4(\Theta=\theta \mid E_1 \cap E_2 \cap E_3^c \cap E_4^c)$$

Perciò

$$\pi_4(\Theta=\theta \mid E_1 \cap E_2 \cap E_3^c \cap E_4^c) = K\, P(E_1 \cap E_2 \cap E_3^c \cap E_4^c \mid \Theta=\theta) \cdot \pi_0(\theta) =$$

$$= K\, P(E_1 \mid \Theta=\theta) \cdot P(E_2 \mid \Theta=\theta) \cdot \left(1-P(E_3 \mid \Theta=\theta)\right) \cdot \left(1-P(E_4 \mid \Theta=\theta)\right) \cdot \pi_0(\theta)$$

$$= K \cdot \theta \cdot \theta \cdot (1-\theta)(1-\theta) \cdot \pi_0(\theta) =$$

$$= K\, \theta^2 \cdot (1-\theta)^2 \cdot \pi_0(\theta)$$

Chi sono K e $\pi_0(\theta)$? Calcoliamoli:

Cominciamo con $\pi_0(\theta)$:

Sappiamo che
$$\pi_0(\theta) \overset{def}{=} \frac{\partial}{\partial \theta} F_\Theta(\theta)$$

Dalla traccia sappiamo che
$$F_\Theta(\theta) = c \int_{-\infty}^{\theta} \eta^2 (1-\eta)\, \mathbb{1}_{(0,1]}(\eta)\, d\theta$$

$\Rightarrow$
$$\pi_0(\theta) = \frac{\partial}{\partial \theta} F_\Theta(\theta) = c\, \theta^2 (1-\theta) \cdot \mathbb{1}_{(0,1]}(\theta)$$

Calcoliamo C:

Deve valere
$$1 = c \int_0^1 \eta^2 (1-\eta)\, d\eta$$

$\Rightarrow$
$$1 = c \int_0^1 \eta^2 - \eta^3\, d\eta$$

$\Rightarrow$
$$1 = c \left(\frac{\eta^3}{3} - \frac{\eta^4}{4} \right) \Big|_0^1$$

$\Rightarrow$
$$1 = c \left(\frac{1}{3} - \frac{1}{4} \right) = c \left(\frac{4-3}{12} \right) = c \frac{1}{12}$$

$\Rightarrow$ $c = 12$

Dunque
$$\pi_0(\theta) = 12\, \theta^2 (1-\theta)$$

Allora

$$\pi_4(\Theta=\theta \mid E_1 \cap E_2 \cap E_3^c \cap E_4^c) = K \cdot \theta^2(1-\theta) \cdot \pi_0(\theta) =$$

$$= K\, \theta^2(1-\theta) \cdot 12\, \theta^2(1-\theta) = K \cdot 12 \cdot \theta^4(1-\theta)^3$$

Calcoliamo K:

Per K deve valere che:

$$1 = \int_R \pi_4(\Theta=\theta \mid E_1 \cap E_2 \cap E_3^c \cap E_4^c)\, d\theta =$$

$$= \int_R K \cdot 12\, \theta^4(1-\theta)^3\, d\theta = \ldots = K \frac{3}{70}$$

$$\Rightarrow 1 = K \frac{3}{70} \quad \Rightarrow K = \frac{70}{3}$$

In conclusione

$$\pi_4 = \frac{70}{3}\, 12\, \theta^4(1-\theta)^3$$

Troviamo $\hat{\theta}$:

Sapendo che

$$\pi_4 = \frac{70}{3} \cdot \theta^4(1-\theta)^3$$

Deriviamo e poniamo uguale a zero:

Risulta

Posso ignorare $\frac{10}{3}$

$$\frac{\partial}{\partial \theta} \Pi_4 = 4\theta^3(1-\theta)^3 + \theta^4 \cdot 3(1-\theta)^2(-1) =$$

$$= 4\theta^3(1-\theta)^3 - 3\theta^4(1-\theta)^2$$

$\Rightarrow$

$$4\theta^3(1-\theta)^3 - 3\theta^4(1-\theta)^2 = 0 \iff$$

$$\iff 4\theta^3(1-\theta)^3 = 3\theta^4(1-\theta)^2$$

$$\iff 1 = \frac{3}{4}\theta \cdot \frac{1}{1-\theta}$$

$$\iff \frac{\theta}{1-\theta} = \frac{4}{3} \iff \frac{4}{3}(1-\theta) = \theta$$

$$\iff \frac{4}{3} - \frac{4}{3}\theta = \theta \iff \theta + \frac{4}{3}\theta = \frac{4}{3}$$

$$\iff \theta\left(1 + \frac{4}{3}\right) = \frac{4}{3}$$

$$\iff \theta = \frac{4}{3} \cdot \frac{1}{1+\frac{4}{3}} = \frac{4}{3} \cdot \frac{1}{\frac{3+4}{3}} = \frac{4}{3} \cdot \frac{3}{7} = \frac{4}{7}$$

$$\Leftrightarrow \theta = \frac{4}{3} \cdot \frac{1}{1+\frac{4}{3}} = \frac{4}{3} \cdot \frac{1}{\frac{3+4}{3}} = \frac{4}{3} \cdot \frac{3}{7} = \frac{4}{7}$$

$$\Rightarrow \bar{\theta} = \frac{4}{7}$$

Appello 31/01/2024

mercoledì 6 marzo 2024 16:18

Es 2

$$f(x) = C x^4$$

Calcolare:

1) C e $F_{X,Y}(x,y)$

2) la matrice

$$\begin{pmatrix} Cov(X,X) & Cov(X,Y) & Cov(X,Z) \\ Cov(Y,X) & Cov(Y,Y) & Cov(Y,Z) \\ Cov(Z,X) & Cov(Z,Y) & Cov(Z,Z) \end{pmatrix}$$

con $Z = \sqrt{X}$

Sol

1) Sapendo che

$$1 = \int_0^1 C x^4 \, dx$$

$$\Rightarrow 1 = C \cdot \frac{1}{5} \Rightarrow C = 5$$

$$\Rightarrow f_X(x) = 5x^4$$

Siccome

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)$$

Calcoliamo $F_X(x)$:

$$\frac{\partial}{\partial x} F_X(x) = f_X(x)$$

$$\Rightarrow \quad F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx = \int_{-\infty}^{x} 5x^4\, dx = x^5 \cdot 1_{(0,1]}(x) + 1_{(1,+\infty)}(x)$$

inoltre

$$X \stackrel{d}{=} Y \Rightarrow F_Y(y) = F_X(y)$$

$$\Rightarrow$$

$$F_{X,Y} = \left(x^5 1_{(0,1]} + 1_{(1,+\infty)}\right)\left(y^5 1_{(0,1]} + 1_{(1,+\infty)}\right)$$

2) $Z = \sqrt{X}$

Ci serve $f_Z(z)$:

$$f_Z(z) = \frac{\partial}{\partial z} F_Z(z)$$

Calcoliamo allora $F_Z(z)$:

$$F_Z(z) = P(Z \leq z) = P(\sqrt{X} \leq z) = P(X \leq z^2) =$$

$$= \int_0^{z^2} f_X(x)\, dx = \int_0^{z^2} 5x^4\, dx = 5 \cdot \frac{x^5}{5} \Big|_0^{z^2} = z^{10}$$

$$\Rightarrow$$

$$f_Z(z) = 10\, z^9$$

Ora:

$$Cov(X,X) = Var(X) = E[X^2] - E^2[X]$$

$$E[X] = \int_0^1 f_X(x) \cdot x = \int_0^1 5x^4 \cdot x \, dx = \frac{5}{6}$$

$$E[X^2] = \int_0^1 f_X(x) \cdot x^2 = \int_0^1 5x^4 \cdot x^2 = \frac{5}{7}$$

$\Rightarrow$

$$Cov(X,X) = \frac{5}{7} - \left(\frac{5}{6}\right)^2 = \frac{5}{7} - \frac{25}{36}$$

$Cov(X,Y) = 0$ perché X e Y sono indipendenti

$$Cov(Y,Y) = Var(Y) = E[Y^2] - E^2[Y]$$

$$E[Y] = \int_0^1 f_Y(y) \cdot y = \int_0^1 5y^4 \cdot y \, dy = \frac{5}{6}$$

$$E[Y^2] = \dots = \frac{5}{7}$$

$\Rightarrow$

$$Cov(Y,Y) = \frac{5}{7} - \frac{25}{36}$$

$Cov(Y,X) = 0$

$Cov(Y,Z) = 0$

$Cov(Z,Y) = 0$

$$Cov(Z,Z) = Var(Z) = E[Z^2] - E^2[Z]$$

$\Rightarrow$

$$E[Z^2] = E[X]$$

$$E[z^2] = E[X]$$

$$E[z] = E[\sqrt{x}] = \int_0^1 f_X(x) \cdot \sqrt{x} = \int_0^1 5x^4 \cdot x^{\frac{1}{2}} =$$

$$= 5 \int_0^1 x^{9/2} \, dx = 5 \cdot \left. \frac{x^{11/2}}{\frac{11}{2}} \right|_0^1 = \frac{10}{11}$$

$\Downarrow$ oppure

$$\int_0^1 f_z(z) \cdot z \, dz$$

$$\Rightarrow \; Cov(z, z) = \frac{5}{6} - \left(\frac{10}{11}\right)^2$$

$$Cov(x, z) = E[xz] - E[x] \cdot E[z]$$

$$\Rightarrow \; E[xz] = E[x\sqrt{x}] = \int_0^1 f_X(x) \cdot x\sqrt{x} =$$

$$= \int_0^1 5x^4 \cdot x \cdot x^{1/2} = 5 \int_0^1 x^{11/2} = 5 \left. \frac{x^{13/2}}{\frac{13}{2}} \right|_0^1 = \frac{10}{13}$$

Es 4

$$F_\Theta(\theta) \overset{df}{=} c \int_{-\infty}^{\theta} y(1-y) \, 1_{[0,1]} \, dy$$

4 osservazioni : $(1, 1, 0, 0)$

Calcolare densità a posteriori e stimatore massima verosimiglianza

Sol

$$\pi_4(\Theta=\theta \mid E_1 \cap E_2 \cap E_3^c \cap E_4^c) = k\, P(E_1 \cap E_2 \cap E_3^c \cap E_4^c \mid \Theta=\theta)\, \pi_0(\theta) =$$

$$= k \cdot \pi_0(\theta) \cdot P(E_1 \mid \Theta=\theta) \cdot P(E_2 \mid \Theta=\theta) \cdot P(E_3^c \mid \Theta=\theta) \cdot P(E_4^c \mid \Theta=\theta) =$$

$$= k \cdot \pi_0(\theta) \cdot \theta \cdot \theta \cdot (1-\theta) \cdot (1-\theta) =$$

$$= k\, \pi_0(\theta) \cdot \theta^2 (1-\theta)^2$$

Dunque

$$\pi_0 = \frac{\partial}{\partial \theta} F_\Theta(\theta) = c\, \theta (1-\theta) \cdot 1_{[0,1]}(\theta)$$

Siccome

$$1 = \int_0^1 c\, y(1-y)\, dy$$

$$\Rightarrow 1 = c \left(\frac{y^2}{2} - \frac{y^3}{3} \right) \Big|_0^1 = c\, \frac{1}{6} \quad \Rightarrow c = 6$$

$$\Rightarrow \pi_0(\theta) = 6\, \theta \cdot (1-\theta)$$

$$\Rightarrow \pi_4(\theta) = k \cdot 6\, \theta (1-\theta)\, \theta^2 (1-\theta)^2 = k\, 6\, \theta^3 (1-\theta)^3$$

Vale che

$$1 = \int_{\mathbb{R}} \pi_4(\theta) = \int_{\mathbb{R}} 6k\, \theta^3 (1-\theta)^3\, d\theta =$$

$$= 6k \int_0^1 \theta^3 (1-\theta)^3\, d\theta =$$

$$= 6k\, \frac{\Gamma(4)\Gamma(4)}{\Gamma(8)} = 6k\, \frac{3!\, 3!}{7!} = 6k\, \frac{36}{7!}$$

$\Rightarrow 1 = 6k \frac{36}{7!} \Rightarrow k = \frac{7!}{6 \cdot 36}$

Dunque

$$\pi_4(\theta) = \frac{7!}{6 \cdot 36} \cdot 6\, \theta^3 (1-\theta)^3$$

Troviamo $\bar{\theta}$:

$$\pi_4(\theta) = \frac{7!}{36} \theta^3 (1-\theta)^3$$

Deriviamo e imponiamo uguale a zero:

$$\frac{\partial}{\partial \theta} \pi_4(\theta) = 3\theta^2 (1-\theta)^3 + \theta^3 \cdot 3 (1-\theta)^2 (-1)$$
$$= 3\theta^2 (1-\theta)^3 - 3\theta^3 (1-\theta)^2$$

$$\Rightarrow 3\theta^2 (1-\theta)^3 - 3\theta^3 (1-\theta)^2 = 0 \iff$$

$$\iff 3\theta^2 (1-\theta)^3 = 3\theta^3 (1-\theta)^2$$

$$\iff 1 = \theta \cdot \frac{1}{1-\theta} \iff 1-\theta = \theta \iff 2\theta = 1 \iff \theta = \frac{1}{2}$$

$$\Rightarrow \bar{\theta} = \frac{1}{2}$$

$\blacksquare$

Es 3

$S = \{1,2,3,4\}$

1) Elencare le classi di equivalenza e calcolarne il periodo

2) Ci sono stati Assorbenti?

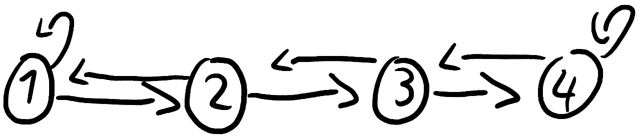
3) Se $\{P_{i,j}\}$ indica la matrice delle probabilità di transizione, calcolare

$$\lim_{n\to\infty} P\{\omega\in\Omega \mid X_n(\omega)=4\}$$

$$\lim_{n\to\infty} \left((P^n)_{1,2} + (P^n)_{3,4} \right)$$

$$\lim_{n\to\infty} (P^{2n})_{23}$$

Sol



$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 3 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 4 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$

Catena aperiodica

2) $\nexists\ i$ t.c. $P_{ii} = 1 \Rightarrow$ Non ci sono stati assorbenti

3) Usiamo il <u>teorema ergodico</u>, cioè

$$\lim_{n \to \infty} (P^n)_{i,j} = \pi_j \qquad \forall\, i,j \in S$$

dove $(\pi_1, \dots, \pi_4)$ è la distribuzione di probabilità stazionaria ovvero tale che $\forall j \in S$ $\sum_{i \in S} \pi_i P_{i,j} = \pi_j$

Per cui

$$\begin{cases} \pi_1 = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 \\ \pi_2 = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_3 \\ \pi_3 = \frac{1}{2}\pi_2 + \frac{1}{2}\pi_4 \\ \pi_4 = \frac{1}{2}\pi_3 + \frac{1}{2}\pi_4 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases}$$

$\Rightarrow$

$$\begin{cases} \frac{1}{2}\pi_1 = \frac{1}{2}\pi_2 \Rightarrow \pi_1 = \pi_2 \\ \frac{1}{2}\pi_2 = \frac{1}{2}\pi_3 \Rightarrow \pi_2 = \pi_3 \\ \frac{1}{2}\pi_3 = \frac{1}{2}\pi_4 \Rightarrow \pi_3 = \pi_4 \\ \frac{1}{2}\pi_4 = \frac{1}{2}\pi_4 \Rightarrow OK! \end{cases}$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \Rightarrow 4\pi_1 = 1 \Rightarrow \pi_1 = \frac{1}{4}$$

$\Rightarrow$ $\pi_1 = \frac{1}{4}$

$\pi_2 = \frac{1}{4}$

$\pi_3 = \frac{1}{4}$

$\pi_4 = \frac{1}{4}$

Possiamo ora calcolare i limiti:

non lo so perché

1) $\lim_{n \to \infty} P\{\omega \in \Omega \mid X_n(\omega) = 4\} = \frac{1}{4}$ (???)

1) $\lim_{n\to\infty} P\{w \in \Omega \mid X_n(w) = 4\} = \frac{1}{4}$ (:':/)

Per il teorema ergodico

2) $\lim_{n\to\infty} \left((P^n)_{1,2} + (P^n)_{3,4} \right) \overset{!}{=} \pi_2 + \pi_4 = \frac{1}{2}$

3) $\lim_{n\to\infty} (P^{2n})_{2,3} = \lim_{n\to\infty} (P^n)_{1,2} = \lim_{n\to\infty} (P^n)_{3,4} = \pi_4 = \frac{1}{4}$

$\Downarrow$ ← Soluzione mia

Sia $K = 2n$

$\Rightarrow \lim_{n\to\infty} (P^{2n})_{2,3} = \lim_{K\to\infty} (P^K)_{2,3} = \pi_3 = \frac{1}{4}$

$\blacksquare$

1) $\lim (P\{w \in \Omega \mid X_n(w) = 4)$

$$P^{(n)} = P^{(0)} P^n$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Appello 23/11/2023

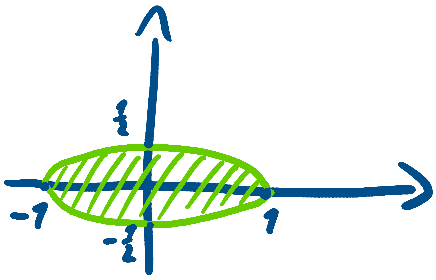
domenica 10 marzo 2024 17:43

Es 2

$$A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + 4y^2 \leq 1\}$$

1) Calcolare densità congiunta e densità marginali
2) X e Y sono indipendenti?
3) Calcolare la densità di $Z = X^2$

SOL



$$A_{mis} = \pi \cdot a \cdot b = \pi \cdot 1 \cdot \frac{1}{2} = \frac{\pi}{2}$$

$$\Rightarrow f(x,y) = \begin{cases} \frac{1}{A} & \text{se } x \in A \\ 0 & \text{altrove} \end{cases} = \begin{cases} \frac{2}{\pi} & \text{se } x \in A \\ 0 & \text{altrove} \end{cases}$$

Dunque

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\, dy =$$

$x^2 + 4y^2 \leq 1$

$\Rightarrow 4y^2 \leq 1 - x^2$

$\Rightarrow y^2 \leq \frac{1-x^2}{4}$

$\Rightarrow -\frac{1}{2}\sqrt{1-x^2} \leq y \leq \frac{1}{2}\sqrt{1-x^2}$

$$= \int_{-\frac{\sqrt{1-x^2}}{2}}^{\frac{\sqrt{1-x^2}}{2}} f(x,y)\, dy =$$

$$= \int_{-\frac{\sqrt{1-x^2}}{2}}^{\frac{\sqrt{1-x^2}}{2}} \frac{2}{\pi} \, dy = \frac{2}{\pi} \cdot \left(\frac{\sqrt{1-x^2}}{2} + \frac{\sqrt{1-x^2}}{2} \right) = \frac{2}{\pi} \sqrt{1-x^2}$$

Analogamente

$$f_Y(y) = \int_{\mathbb{R}} f(x,y) \, dx =$$

$x^2 + 4y^2 \leq 1$

$\Rightarrow x^2 \leq 1 - 4y^2$

$\Rightarrow$

$-\sqrt{1-4y^2} \leq x \leq \sqrt{1-4y^2}$

$$= \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} f(x,y) \, dx =$$

$$= \frac{2}{\pi} \left(\sqrt{1-4y^2} + \sqrt{1-4y^2} \right) = \frac{4}{\pi} \sqrt{1-4y^2}$$

Riassumendo:

$$f(x,y) = \begin{cases} \frac{2}{\pi} \\ 0 \end{cases}$$

$$f_X(x) = \frac{2}{\pi} \sqrt{1-x^2}$$

$$f_Y(y) = \frac{4}{\pi} \sqrt{1-4y^2}$$

Segue che:

$$E[X] = \int_{\mathbb{R}} f_X(x) \cdot x \, dx = \int_{-1}^{1} \frac{2}{\pi} \sqrt{1-x^2} \cdot x \, dx =$$

$$= \frac{2}{\pi} \int_{-1}^{1} x\sqrt{1-x^2} \, dx = \frac{2}{\pi} \left(-\frac{1}{3} (1-x^2) \right) \Big|_{-1}^{1} =$$

$$= \frac{2}{\pi}\left(-\frac{1}{3}(1-1) - \left(-\frac{1}{3}(1-1)\right)\right) = 0$$

$$E[Y] = \int_{R} f_Y(y)\cdot y \, dy = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{4}{\pi}\sqrt{1-4y^2}\cdot y \, dy =$$

= "è inutile calcolarlo"

$$E[XY] = \iint_D f(x,y)\, xy \, dx\, dy = \int_{-1}^{1}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{2}{\pi} xy \, dy\, dx =$$

$$= \frac{2}{\pi}\int_{-1}^{1} x \left.\frac{y^2}{2}\right|_{-\frac{1}{2}}^{1/2} dx = \frac{2}{\pi}\int_{-1}^{1} x\left(\frac{1}{2}\cdot\frac{1}{4} - \frac{1}{2}\frac{1}{4}\right) dx = 0$$

$\Rightarrow$ $E[XY] = 0$

$E[X]\cdot E[Y] = 0$

$\Rightarrow$ $E[XY] = E[X]\,E[Y]$

$\Rightarrow$ X e Y sono <u>non</u> correlate

non sono indipendenti

3) Sia $Z = X^2$

Sappiamo che

$$f_X(x) = \frac{2}{\pi}\sqrt{1-x^2}$$

sappiamo che

$$f_X(x) = \frac{2}{\pi}\sqrt{1-x^2}$$

e sappiamo che vale

$$\frac{\partial}{\partial x} F_X = f_X$$

A noi interessa calcolare $f_Z(z)$

$$F_Z(z) = P(Z \leq z) = P(X^2 \leq z) =$$

$$= P(-\sqrt{z} \leq X \leq \sqrt{z}) =$$

$$= P(X \leq \sqrt{z}) - P(X < -\sqrt{z}) =$$

$$= F_X(\sqrt{z}) - F_X(-\sqrt{z})$$

Dunque dalla relazione

$$\frac{\partial}{\partial x} F_X = f_X$$

segue che:

$$\frac{\partial}{\partial z} F_Z = f_Z$$

$\Rightarrow$

$$f_Z = \frac{\partial}{\partial z} F_Z(z) = \frac{\partial}{\partial z}\left(F_X(\sqrt{z}) - F_X(-\sqrt{z})\right) =$$

$$= \frac{\partial}{\partial z} F_X(\sqrt{z}) - \frac{\partial}{\partial z} F_X(-\sqrt{z}) =$$

$$= f_X(\sqrt{z}) \cdot \frac{1}{2\sqrt{z}} - f_X(-\sqrt{z}) \cdot \left(-\frac{1}{2\sqrt{z}}\right) =$$

$$= \frac{1}{2\sqrt{z}} \cdot f_X(\sqrt{z}) + \frac{1}{2\sqrt{z}} f(-\sqrt{z}) =$$

$$= \frac{1}{2\sqrt{z}} \left(f_X(\sqrt{z}) + f_X(-\sqrt{z}) \right) =$$

$$= \frac{1}{2\sqrt{z}} \left(\frac{2}{\pi}\sqrt{1-z} + \frac{2}{\pi}\sqrt{1-z} \right) = \frac{2}{\pi\sqrt{z}} \sqrt{1-z} =$$

$$= \frac{2}{\pi} \sqrt{\frac{1-z}{z}}$$

∎

Es 3

$$f(x|\theta) = \frac{\theta}{2} e^{-\theta|x|}$$

Calcolare $\bar{\theta}$

SOL

$$\mathcal{L}(\theta | x_1, \dots, x_n) = \prod_{i=1}^{n} f(x_i|\theta)$$

$\Rightarrow$

$$\mathcal{L}(\theta | x_1, \dots, x_n) = \prod_{i=1}^{n} \frac{\theta}{2} e^{-\theta|x_i|}$$

Applicando il logaritmo si ha:

Applichiamo il logaritmo si ha

$$\log(\mathcal{L}(\theta \mid x_1, \dots x_n)) = \log\left(\prod_{i=1}^{n} \frac{\theta}{2} e^{-\theta |x_i|}\right) =$$

$$= \sum_{i=1}^{n} \log\left(\frac{\theta}{2} e^{-\theta |x_i|}\right) =$$

$$= \sum_{i=1}^{n} \log(\theta) - \log(2) + \log\left(e^{-\theta |x_i|}\right) =$$

$$= n \log(\theta) - n \log(2) + \sum_{i=1}^{n} -\theta |x_i|$$

Derivando e imponendo l'uguaglianza a zero si ha:

$$n \frac{1}{\theta} - \sum_{i=1}^{n} |x_i| = 0$$

$$\Rightarrow \frac{n}{\theta} = \sum_{i=1}^{n} |x_i| \quad \Rightarrow \theta = \frac{n}{\sum_{i=1}^{n} |x_i|}$$

$$\Rightarrow \bar{\theta} = \frac{n}{\sum_{i=1}^{n} |x_i|}$$

∎

ES 13.1 (Libro Baggini)

Una catena di Markov $(X_n)_{n\in\mathbb{N}}$ con insieme degli stati $S=\{1,2,3,4\}$ ha la seguente matrice di transizione

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix} \end{array}
$$

e distribuzione iniziale

$$\mu(1)=\mu(2)=\mu(3)=\mu(4)=\frac{1}{4}$$

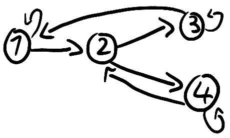
a) Dire quali sono le classi di equivalenza fra stati ed i loro periodi

b) Calcolare $p^{(2)}_{2,1}$, $p^{(2)}_{1,4}$, $p^{(3)}_{1,1}$

c) Calcolare $\lim_{n\to\infty} p^{(n)}_{1,3}$ e $\lim_{n\to\infty} P(X_n=2)$

SOL

a) Per individuare le classi di equivalenza fra gli stati ed il loro periodo, si può costruire un grafico



Dal grafico si deduce che tutti gli stati comunicano fra loro $\Rightarrow$ Esiste un'unica classe di equivalenza.

La catena è aperiodica

b) Cominciamo col calcolare $p^{(2)}_{2,1}$:

$p^{(2)}_{2,1}$ è la probabilità di transitare dallo stato 2 allo stato 1 in 2 passi

Utilizziamo il fatto che tale probabilità da 2 a 1 si può "spezzare" nella somma di tutti i possibili percorsi da 2 a 1, cioè

$$p^{(2)}_{2,1} = \sum_{i\in S} p^{(1)}_{2,i}\, p^{(1)}_{i,1}$$

Si ottiene che

Si ottiene che

$$p^{(2)}_{2,1} = p^{(1)}_{2,1}\, p^{(1)}_{1,1} + p^{(1)}_{2,2} \cdot p^{(1)}_{2,1} + p^{(1)}_{2,3}\, p^{(1)}_{3,1} + p^{(1)}_{2,4}\, p^{(1)}_{4,1} =$$

$$= 0 \cdot \frac{1}{4} + 0 \cdot 0 + \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot 0 = \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}$$

OPPURE: Possiamo ottenere $p^{(2)}_{2,1}$ in un altro modo:

$$p^{(2)}_{2,1} = P_2 \cdot P^1$$

come prodotto "riga per colonna" dove P_2 denota la seconda riga e P^1 denota la prima colonna della matrice P:

Dunque essendo

$$P = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$$

$\Rightarrow$

$$\begin{pmatrix} 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{4} \\ 0 \end{pmatrix} = 0 \cdot \frac{1}{4} + 0 \cdot 0 + \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot 0 = \frac{1}{6}$$

Analogamente si calcolano $p^{(2)}_{1,4}$ e $p^{(2)}_{1,1}$

c)

$$\lim_{n \to \infty} \left(p^{n}_{1,3}\right) \overset{\text{TEO ERGODICO}}{=} \pi_3$$

Calcoliamo allora π_3:

Dobbiamo costruire il sistema

$$\begin{cases} \pi_1 = \frac{1}{4}\pi_1 + \frac{1}{4}\pi_3 \\ \pi_2 = \frac{3}{4}\pi_1 + \frac{1}{3}\pi_4 \\ \pi_3 = \frac{2}{3}\pi_2 + \frac{3}{4}\pi_3 \\ \pi_4 = \frac{1}{3}\pi_2 + \frac{2}{3}\pi_4 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{3}{4}\pi_1 = \frac{1}{4}\pi_3 \Rightarrow \pi_1 = \frac{1}{3}\pi_3 \\ \pi_2 = \frac{3}{4}\pi_1 + \frac{1}{3}\pi_4 \\ \frac{1}{4}\pi_3 = \frac{2}{3}\pi_2 \Rightarrow \pi_3 = \frac{8}{3}\pi_2 \\ \frac{1}{3}\pi_4 = \frac{1}{3}\pi_2 \Rightarrow \pi_4 = \pi_2 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases} \Rightarrow \begin{cases} \pi_1 = \frac{1}{3}\pi_3 \\ \pi_2 = \frac{1}{4}\pi_3 + \frac{1}{3}\pi_4 \\ \pi_3 = \frac{8}{3}\pi_2 \\ \pi_4 = \pi_2 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \pi_1 = \frac{1}{3}\pi_3 \\ \pi_3 = \frac{8}{3}\pi_2 \\ \pi_2 = \frac{2}{3}\pi_2 + \frac{1}{3}\pi_2 = \pi_2 \Rightarrow ok! \\ \pi_4 = \pi_2 \\ \frac{1}{3}\pi_3 + \pi_2 + \pi_3 + \pi_2 = 1 \end{cases}$$

$$\Rightarrow \frac{1}{3}\pi_3 + \pi_3 + 2\pi_2 = 1$$

$$\Rightarrow \frac{4}{3}\pi_3 + 2 \cdot \frac{3}{8}\pi_3 = 1 \quad \Rightarrow \frac{4}{3}\pi_3 + \frac{3}{4}\pi_3 = 1$$

$\frac{1}{3}\pi_3 + \pi_3 + \dots$

$\Rightarrow \frac{4}{3}\pi_3 + 2 \cdot \frac{3}{8}\pi_3 = 1 \quad \Rightarrow \quad \frac{4}{3}\pi_3 + \frac{3}{4}\pi_3 = 1$

$\Rightarrow \frac{25}{12}\pi_3 = 1 \quad \Rightarrow \quad \pi_3 = \frac{12}{25}$

Per cui

$$\lim_{n\to\infty} \left(p_{1,3}^{n}\right) = \pi_3 = \frac{12}{25}$$

Calcoliamo ora $\lim_{n\to\infty} P(X_n = 2)$:

Osserviamo che

(Sarebbero le condizioni **INIZIALI** $\mu_1, \mu_2, \mu_3, \mu_4$ ↓ $P(X_0 = i)$)

$$P(X_n=2) = \sum_{i=1}^{4} P(X_n = 2 \mid X_0 = i) \cdot P(X_0 = i) =$$

$$= \frac{1}{4} \sum_{i=1}^{4} P(X_n = 2 \mid X_0 = i) = \frac{1}{4} \sum_{i=1}^{4} p_{i,2}^{n}$$

Perché per ogni i vale che:

$$\lim_{n\to\infty} p_{i,2}^{(n)} = \pi_2$$

si ottiene

$$P(X_n=2) = \lim_{n\to\infty} \frac{1}{4} \cdot \sum_{i=1}^{4} p_{i,2}^{(n)} = \frac{1}{4} \cdot 4\pi_2 = \pi_2 = \frac{3}{8}\pi_3$$

$$= \frac{9}{50}$$

ES 13.2

Una catena di Markov X_n, $n = 0,1,2,\dots$ con insieme degli stati

$$S = \{1,2,3,4,5,6\}$$

ha la seguente matrice di transizione

	1	2	3	4	5	6
1	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$
2	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0
3	0	$\frac{1}{3}$	0	$\frac{2}{3}$	0	0
4	0	0	$\frac{2}{3}$	0	$\frac{1}{3}$	0
5	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$
6	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0

e distribuzione iniziale

$\mu(1) = \frac{1}{3}$

$\mu(2) = \frac{2}{3}$

$\mu(3) = \mu(4) = \mu(5) = \mu(6) = 0$

1) Dire quali sono le classi di equivalenze fra stati e i loro periodi

2) Calcolare : $\lim_{n \to \infty} P_{1,5}^{(2n)}$

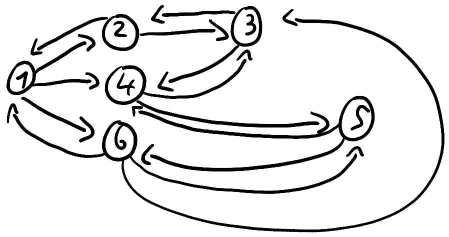
$\lim_{n \to \infty} P_{3,5}^{(n)}$

$\lim_{n \to \infty} P_{2,5}^{(2n)}$

$\lim_{n \to \infty} P(X_n = 5)$

3) Calcolare $P(X_2 < 3)$

SOL



Tutti gli stati comunicano tra di loro

$\Rightarrow$ Esiste una sola classe di equivalenza

Inoltre il numero di passi che si deve fare da uno stato per tornare nello stesso stato è sempre pari

$\Rightarrow$ Ne segue che il periodo d è pari a 2

$\Rightarrow d = 2$ ← Periodo

$d = MCD(A^+_{5,5})$

dove $A^+_{5,5} = \{n \mid P^{(n)}_{5,5} > 0\}$

2) Per studiare i limiti si considerano le classi di equivalenza della matrice P^2; esse sono due, ciascuna di periodo 1.

Per ottenere la loro composizione non è necessario calcolare esplicitamente tutta la matrice P^2; per esempio la classe di 1 sarà formata da tutti gli stati che comunicano con 1 in un numero pari di passi.

Si ottiene

$[1] = \{1, 3, 5\}$

$[2] = \{2, 4, 6\}$

Poiché 2 e 5 **non** comunicano in un numero pari di passi si ottiene che

$$p_{2,5}^{(2n)} = 0$$

$$\Rightarrow \lim_{n\to\infty} p_{2,5}^{(2n)} = 0$$

Passiamo ora al calcolo degli altri limiti:

Prima di tutto ci serviranno $\pi_1, \pi_2, \dots, \pi_6$

$$\begin{cases} \pi_1 = \frac{1}{2}\pi_2 + \frac{1}{3}\pi_6 \\ \pi_2 = \frac{1}{3}\pi_1 + \frac{2}{3}\pi_3 \\ \pi_3 = \frac{1}{2}\pi_2 + \frac{2}{3}\pi_4 + \frac{1}{3}\pi_6 \\ \pi_4 = \frac{1}{3}\pi_1 + \frac{2}{3}\pi_3 + \frac{1}{2}\pi_5 \\ \pi_5 = \frac{1}{3}\pi_4 + \frac{1}{3}\pi_6 \\ \pi_6 = \frac{1}{3}\pi_1 + \frac{1}{2}\pi_5 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \pi_1 = \frac{1}{2}\pi_2 + \frac{1}{3}\pi_6 = \frac{1}{2}\left(\frac{2}{15}\pi_6 + \frac{2}{5}\pi_3\right) + \frac{1}{3}\pi_6 = \frac{1}{15}\pi_6 + \frac{1}{5}\pi_3 + \frac{1}{3}\pi_6 = \frac{2}{5}\pi_6 + \frac{1}{5}\pi_3 = \frac{2}{5}\pi_6 + \frac{1}{5}\left(\frac{1}{2}\pi_6 + \frac{5}{6}\pi_4\right) = \frac{2}{5}\pi_6 + \frac{1}{10}\pi_6 + \frac{1}{6}\pi_4 = \frac{1}{2}\pi_6 + \frac{1}{6}\pi_4 \\ \pi_2 = \frac{1}{3}\left(\frac{1}{2}\pi_2 + \frac{1}{3}\pi_6\right) + \frac{2}{3}\pi_3 = \frac{1}{6}\pi_2 + \frac{1}{9}\pi_6 + \frac{1}{3}\pi_3 \Rightarrow \frac{5}{6}\pi_2 = \frac{1}{9}\pi_6 + \frac{1}{3}\pi_3 \Rightarrow \pi_2 = \frac{2}{15}\pi_6 + \frac{2}{5}\pi_3 = \frac{2}{15}\pi_6 + \frac{2}{5}\left(\frac{1}{2}\pi_6 + \frac{5}{6}\pi_4\right) = \frac{2}{15}\pi_6 + \frac{1}{5}\pi_6 + \frac{1}{3}\pi_4 = \frac{1}{3}\pi_6 + \frac{1}{3}\pi_4 \\ \pi_3 = \frac{1}{2}\left(\frac{2}{15}\pi_6 + \frac{2}{5}\pi_3\right) + \frac{2}{3}\pi_4 + \frac{1}{3}\pi_6 = \frac{1}{15}\pi_6 + \frac{1}{5}\pi_3 + \frac{2}{3}\pi_4 + \frac{1}{3}\pi_6 = \frac{2}{5}\pi_6 + \frac{1}{5}\pi_3 + \frac{2}{3}\pi_4 \Rightarrow \frac{4}{5}\pi_3 = \frac{2}{5}\pi_6 + \frac{2}{3}\pi_4 \Rightarrow \pi_3 = \frac{1}{2}\pi_6 + \frac{5}{6}\pi_4 \\ \pi_4 = \frac{1}{3}\left(\frac{1}{2}\pi_6 + \frac{1}{6}\pi_4\right) + \frac{2}{3}\left(\frac{1}{2}\pi_6 + \frac{5}{6}\pi_4\right) + \frac{1}{2}\pi_5 = \frac{1}{6}\pi_6 + \frac{1}{18}\pi_4 + \frac{1}{3}\pi_6 + \frac{5}{9}\pi_4 + \frac{1}{2}\pi_5 = \frac{1}{2}\pi_6 + \frac{11}{18}\pi_4 + \frac{1}{2}\pi_5 \Rightarrow \frac{7}{18}\pi_4 = \frac{1}{2}\pi_6 + \frac{1}{2}\pi_5 \Rightarrow \pi_4 = \frac{9}{7}\pi_6 + \frac{9}{7}\pi_5 \\ \pi_5 = \frac{1}{3}\left(\frac{9}{7}\pi_6 + \frac{9}{7}\pi_5\right) + \frac{1}{3}\pi_6 = \frac{3}{7}\pi_6 + \frac{3}{7}\pi_5 + \frac{1}{3}\pi_6 = \frac{16}{21}\pi_6 + \frac{3}{7}\pi_5 \Rightarrow \frac{4}{7}\pi_5 = \frac{16}{21}\pi_6 \Rightarrow \pi_5 = \frac{4}{3}\pi_6 \end{cases}$$

$$\pi_6 = \frac{1}{3}\left(\frac{1}{2}\pi_6 + \frac{1}{6}\pi_4\right) + \frac{1}{2}\left(\frac{4}{3}\pi_6\right) = \frac{1}{6}\pi_6 + \frac{1}{18}\pi_4 + \frac{2}{3}\pi_6 = \frac{1}{6}\pi_6 + \frac{2}{3}\pi_6 + \frac{1}{18}\left(\frac{9}{7}\pi_6 + \frac{9}{7}\pi_5\right) =$$

$$= \frac{1}{6}\pi_6 + \frac{2}{3}\pi_6 + \frac{1}{14}\pi_6 + \frac{1}{14}\left(\frac{4}{3}\pi_6\right) = \frac{1}{6}\pi_6 + \frac{2}{3}\pi_6 + \frac{1}{14}\pi_6 + \frac{2}{21}\pi_6 = \pi_6 \Rightarrow OK!$$

$$\Rightarrow \begin{cases} \pi_6 = \pi_6 \\ \pi_1 = \frac{1}{2}\pi_6 + \frac{1}{6}\pi_4 = \frac{1}{2}\pi_6 + \frac{1}{6}\left(\frac{9}{7}\pi_6 + \frac{9}{7}\pi_5\right) = \frac{3}{14}\pi_6 + \frac{3}{14}\pi_5 = \frac{3}{14}\pi_6 + \frac{3}{14}\left(\frac{4}{3}\pi_6\right) = \frac{2}{7}\pi_6 + \frac{3}{14}\pi_6 = \frac{1}{2}\pi_6 \\ \pi_2 = \frac{1}{3}\pi_6 + \frac{1}{3}\pi_4 = \frac{1}{3}\pi_6 + \frac{1}{3}\left(\frac{9}{7}\pi_6 + \frac{9}{7}\pi_5\right) = \frac{1}{3}\pi_6 + \frac{3}{7}\pi_6 + \frac{3}{7}\pi_5 = \frac{1}{3}\pi_6 + \frac{3}{7}\pi_6 + \frac{3}{7}\left(\frac{4}{3}\pi_6\right) = \frac{4}{3}\pi_6 \\ \pi_3 = \frac{1}{2}\pi_6 + \frac{5}{6}\pi_4 = \frac{1}{2}\pi_6 + \frac{5}{6}\left(\frac{9}{7}\pi_6 + \frac{9}{7}\pi_5\right) = \frac{1}{2}\pi_6 + \frac{15}{14}\pi_6 + \frac{15}{14}\pi_5 = \frac{1}{2}\pi_6 + \frac{15}{14}\pi_6 + \frac{15}{14}\left(\frac{4}{3}\pi_6\right) = 3\pi_6 \\ \pi_4 = \frac{9}{7}\pi_6 + \frac{9}{7}\pi_5 = \frac{9}{7}\pi_6 + \frac{9}{7}\left(\frac{4}{3}\pi_6\right) = 3\pi_6 \\ \pi_5 = \frac{4}{3}\pi_6 \end{cases}$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 = 1$$

$$\Rightarrow \frac{1}{2}\pi_6 + \frac{4}{3}\pi_6 + 3\pi_6 + 3\pi_6 + \frac{4}{3}\pi_6 + \pi_6 = 1$$

$$\Rightarrow \frac{61}{6}\pi_6 = 1 \Rightarrow \pi_6 = \frac{6}{61}$$ (purtroppo non coincide col risultato del libro)

METODO LIBRO: Restano da calcolare i seguenti limiti:

$$\lim_{n\to\infty} p_{1,5}^{(2n)}$$

$$\lim_{n\to\infty} p_{3,5}^{(n)}$$

$$\lim_{n\to\infty} P(X_n = 5)$$

Cominciamo con

$$\lim_{n\to\infty} P_{1,5}^{(2n)}$$

Ci serve la matrice P^2:

$$P^2 = P \cdot P = \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{5}{18} & 0 & \frac{1}{2} & 0 & \frac{2}{9} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{11}{18} & 0 & \frac{2}{9} & 0 \\ 0 & \frac{2}{9} & 0 & \frac{11}{18} & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{2} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{2}{9} & 0 & \frac{1}{2} & 0 & \frac{5}{18} \end{pmatrix}$$

Abbiamo detto prima che le classi di equivalenza sono

$$[1] = \{1, 3, 5\}$$
$$[2] = \{2, 4, 6\}$$

Lo stato $5 \in [1]$ consideriamo allora la sottomatrice di P^2 di elementi in posizione $(1,1), (1,3), (1,5)$
$(3,1), (3,3), (3,5)$
$(5,1), (5,3), (5,5)$

$$\Rightarrow \begin{pmatrix} \frac{5}{18} & \frac{1}{2} & \frac{2}{9} \\ \frac{1}{6} & \frac{11}{18} & \frac{2}{9} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

Risulta che

$$\lim_{n\to\infty} P_{1,5}^{(2n)} = \pi_5$$

Calcoliamo π_5:

$$\begin{cases} \pi_1 = \frac{5}{18}\pi_1 + \frac{1}{6}\pi_3 + \frac{1}{6}\pi_5 \\ \pi_3 = \frac{1}{2}\pi_1 + \frac{11}{18}\pi_3 + \frac{1}{2}\pi_5 \\ \pi_5 = \frac{2}{9}\pi_1 + \frac{2}{9}\pi_3 + \frac{1}{3}\pi_5 \\ \pi_1 + \pi_3 + \pi_5 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{13}{18}\pi_1 = \frac{1}{6}\pi_3 + \frac{1}{6}\pi_5 \Rightarrow \pi_1 = \frac{3}{13}\pi_3 + \frac{3}{13}\pi_5 \Rightarrow \pi_1 = \frac{3}{13} \cdot \frac{9}{4}\pi_5 + \frac{3}{13}\pi_5 = \frac{3}{4}\pi_5 \\ \pi_3 = \frac{1}{2}\left(\frac{3}{13}\pi_3 + \frac{3}{13}\pi_5\right) + \frac{11}{18}\pi_3 + \frac{1}{2}\pi_5 = \frac{3}{26}\pi_3 + \frac{3}{26}\pi_5 + \frac{11}{18}\pi_3 + \frac{1}{2}\pi_5 = \frac{85}{117}\pi_3 + \frac{8}{13}\pi_5 \Rightarrow \frac{32}{117}\pi_3 = \frac{8}{13}\pi_5 \Rightarrow \pi_3 = \frac{9}{4}\pi_5 \\ \pi_5 = \frac{2}{9} \cdot \frac{3}{4}\pi_5 + \frac{2}{9} \cdot \frac{9}{4}\pi_5 + \frac{1}{3}\pi_5 = \pi_5 \Rightarrow OK \end{cases}$$

$$\frac{3}{4}\pi_5 + \frac{9}{4}\pi_5 + \pi_5 = 1 \Rightarrow 4\pi_5 = 1 \Rightarrow \pi_5 = \frac{1}{4}$$

Dunque
$$\lim_{n\to\infty} p_{1,5}^{(2n)} = \pi_5 = \frac{1}{4}$$

Calcoliamo ora
$$\lim_{n\to\infty} p_{3,5}^{(n)}$$

Se n è PARI : Sia $n = 2K$, allora
$$\lim_{n\to\infty} p_{3,5}^{(n)} = \lim_{K\to\infty} p_{3,5}^{(2K)} = \pi_5 = \frac{1}{4}$$

Se n è DISPARI : Sia $n = 2K+1$, allora
$$\lim_{n\to\infty} p_{3,5}^{(n)} = \lim_{K\to\infty} p_{3,5}^{(2K+1)} = 0$$
in quanto non è possibile passare dallo stato 3 allo stato 5 con un numero dispari di passi

Infine calcoliamo
$$\lim_{n\to\infty} P(X_n = 5)$$

Si ha che:
$$P(X_n=5) = \sum_{i=1}^{6} P(X_n=5 / X_0=i) \cdot P(X_0=i) =$$
$$= \frac{1}{3} P(X_n=5/X_0=1) + \frac{2}{3} P(X_n=5/X_0=2) =$$
$$= \frac{1}{3} p_{1,5}^{(n)} + \frac{2}{3} p_{2,5}^{(n)}$$

Se n è <u>PARI</u> : Sia $n = 2K$
$$\Rightarrow \frac{1}{3} p_{1,5}^{(2K)} + \frac{2}{3} p_{2,5}^{(2K)} = \frac{1}{3} p_{1,5}^{(2K)}$$
$$\Rightarrow P(X_n=5) = \frac{1}{3} p_{1,5}^{(2K)} \xrightarrow{K\to\infty} \frac{1}{3} \cdot \pi_5 = \frac{1}{12}$$

Se n è <u>DISPARI</u> : Sia $n = 2K+1$
$$\Rightarrow \frac{1}{3} p_{1,5}^{(2K+1)} + \frac{2}{3} p_{2,5}^{(2K+1)} =$$
$$= \frac{2}{3} p_{2,5}^{(2K+1)} = \frac{2}{3} \sum_{i=1}^{6} p_{2,i}^{(1)} p_{i,5}^{(2K)} =$$
$$= \frac{2}{3}\left(p_{2,1}^{(1)} p_{1,5}^{(2K)} + p_{2,2}^{(1)} p_{2,5}^{(2K)} + p_{2,3}^{(1)} p_{3,5}^{(2K)} + p_{2,4}^{(1)} p_{4,5}^{(2K)} + p_{2,5}^{(1)} p_{5,5}^{(2K)} + p_{2,6}^{(1)} p_{6,5}^{(2K)} \right) =$$
$$= \frac{2}{3}\left(\frac{1}{2} \cdot \pi_5 + 0 + \frac{1}{2} \cdot \pi_5 + 0 + 0 + 0 \right) =$$

$= \frac{2}{3}\left(p_{2,1}\, p_{1,5} + p_{2,2}\, p_{2,5} + p_{2,3}\, p_{3,5} + p_{2,4}\, p_{4,5} + p_{2,5}\, p_{5,5} + p_{2,6}\, p_{6,5} \right)$

$= \frac{2}{3}\left(\frac{1}{2} \cdot \pi_5 + 0 + \frac{1}{2} \cdot \pi_5 + 0 + 0 + 0 \right) =$

$= \frac{2}{3}\pi_5 = \frac{2}{3} \cdot \frac{1}{4} = \frac{2}{12} = \frac{1}{6}$

3) Per calcolare $P(X_2 < 3)$ ci basta osservare che:

$$P(X_2 < 3) = \sum_{i=1}^{2} P(X_2 = i)$$

$\Rightarrow$

$$P(X_2 < 3) = P(X_2 = 1) + P(X_2 = 2)$$

Perciò

$$P(X_2 = 1) = \sum_{i=1}^{6} P(X_2 = 1 \mid X_0 = i)\, P(X_0 = i) =$$

$$= P(X_2 = 1 \mid X_0 = 1)\, \frac{1}{3} + P(X_2 = 1 \mid X_0 = 2) \cdot \frac{2}{3} =$$

$$= \frac{1}{3}\, p_{1,1}^{(2)} + \frac{2}{3}\, p_{2,1}^{(2)} = \frac{1}{3} \sum_{i=1}^{6} p_{1,i}^{(1)}\, p_{i,1}^{(1)} + \frac{2}{3} \sum_{i=1}^{6} p_{2,i}^{(1)}\, p_{i,1}^{(1)} =$$

$$= \frac{1}{3}\left(p_{1,1}\, p_{1,1} + p_{1,2}\, p_{2,1} + p_{1,3}\, p_{3,1} + p_{1,4}\, p_{4,1} + p_{1,5}\, p_{5,1} + p_{1,6}\, p_{6,1} \right) +$$

$$+ \frac{2}{3}\left(p_{2,1}\, p_{1,1} + p_{2,2}\, p_{2,1} + p_{2,3}\, p_{3,1} + p_{2,4}\, p_{4,1} + p_{2,5}\, p_{5,1} + p_{2,6}\, p_{6,1} \right) =$$

$$= \frac{1}{3}\left(0 + \frac{1}{3} \cdot \frac{1}{2} + 0 + \frac{1}{3} \cdot 0 + 0 + \frac{1}{3} \cdot \frac{1}{3} \right) + \frac{2}{3}\left(\frac{1}{2} \cdot 0 + 0 + \frac{1}{2} \cdot 0 + 0 + 0 + 0 \right) =$$

$$= \frac{1}{3}\left(\frac{1}{6} + \frac{1}{9} \right) = \frac{5}{54}$$

$$P(X_2 = 2) = \sum_{i=1}^{6} P(X_2 = 2 \mid X_0 = i)\, P(X_0 = i) =$$

$$= \frac{1}{3}\, P(X_2 = 2 \mid X_0 = 1) + \frac{2}{3}\, P(X_2 = 2 \mid X_0 = 2) =$$

$$= \frac{1}{3}\, p_{1,2}^{(2)} + \frac{2}{3}\, p_{2,2}^{(2)} =$$

$$= \frac{1}{3} \sum_{i=1}^{6} p_{1,i}^{(1)}\, p_{i,2}^{(1)} + \frac{2}{3} \sum_{i=1}^{6} p_{2,i}^{(1)}\, p_{i,2}^{(1)} =$$

$$\frac{1}{3}\left(\cancel{p_{1,1}\, p_{1,2}} + \cancel{p_{1,2}\, p_{2,2}} + \cancel{p_{1,3}\, p_{3,2}} + \cancel{p_{1,4}\, p_{4,2}} + \cancel{p_{1,5}\, p_{5,2}} + \cancel{p_{1,6}\, p_{6,2}} \right) +$$

$$+ \frac{2}{3}\left(p_{2,1}\, p_{1,2} + \cancel{p_{2,2}\, p_{2,2}} + p_{2,3}\, p_{3,2} + \cancel{p_{2,4}\, p_{4,2}} + \cancel{p_{2,5}\, p_{5,2}} + \cancel{p_{2,6}\, p_{6,2}} \right) =$$

$$= \frac{2}{3}\left(\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} \right) = \frac{2}{9}$$

Dunque

$$P(X_2 < 3) = P(X_2 = 1) + P(X_2 = 2) =$$

$$= \frac{5}{54} + \frac{2}{9}$$

∎

ES 13.3

Una catena di Markov X_n, $n = 0, 1, 2, \dots$, con insieme degli stati $S = \{1, 2, 3, 4\}$ ha la seguente matrice di transizione

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{6} & 0 & 0 & \frac{5}{6} \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 \end{pmatrix} \end{array}$$

e distribuzione iniziale

$\mu(1) = \frac{1}{3}$

$\mu(2) = \frac{1}{3}$

$\mu(3) = \frac{1}{3}$

$\mu(4) = 0$

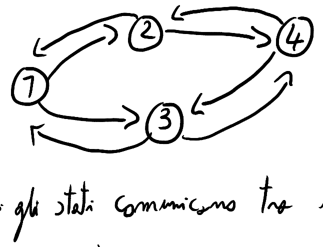
1) Dire quali sono le classi di equivalenza e i loro periodi

2) Calcolare $P(X_5 = 2 \mid X_2 = 3)$, $p_{1,4}^{(2)}$, $P(X_2)$

3) Calcolare: $\lim_{n\to\infty} p_{1,3}^{(2n)}$, $\lim_{n\to\infty} p_{1,4}^{(2n)}$, $\lim_{n\to\infty} p_{2,3}^{(n)}$, $\lim_{n\to\infty} P(X_n = 2)$

Sol

1)



Tutti gli stati comunicano tra loro $\Rightarrow \exists!$ classe di equivalenza

Periodo $d = 2$

$\Rightarrow$ calcolo subito P^2

$$P^2 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{6} & 0 & 0 & \frac{5}{6} \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{6} & 0 & 0 & \frac{5}{6} \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{12} & 0 & 0 & \frac{7}{12} \\ 0 & \frac{7}{12} & \frac{5}{12} & 0 \\ 0 & \frac{17}{24} & \frac{7}{24} & 0 \\ \frac{13}{24} & 0 & 0 & \frac{11}{24} \end{pmatrix}$$

$[1] = \{1, 4\}$

$[2] = \{2, 3\}$

$[2] = \{2, 3\}$

2) • $P(X_5 = 2 \mid X_2 = 3) = P^{(3)}_{3,2} = 0$

Prendo l'elemento in posizione (1,4) della matrice P^2

• $P^{(2)}_{1,4} = \frac{7}{72}$

• $P(X_2)$:

Per calcolare $P(X_2)$ usiamo la formula della distribuzione discreta e quella delle probabilità totali.

Per cui

$$P(X_2) = \sum_{i=1}^{4} i \; P(X_2 = i) = \sum_{i=1}^{4} i \sum_{j=1}^{4} P(X_2 = i \mid X_0 = j) P(X_0 = j)$$

$P(X_0 = j) = \frac{1}{3}$ per $j = 1,2,3$ ed è pari a zero per $j = 4$

$$= \sum_{i=1}^{4} i \cdot \frac{1}{3} \sum_{j=1}^{4} P(X_2 = i \mid X_0 = j) =$$

$$= \sum_{i=1}^{4} i \cdot \frac{1}{3} \left(P(X_2 = i \mid X_0 = 1) + P(X_2 = i \mid X_0 = 2) + P(X_2 = i \mid X_0 = 3) \right)$$

$$= \sum_{i=1}^{4} i \cdot \frac{1}{3} \left(P^{(2)}_{1,i} + P^{(2)}_{2,i} + P^{(2)}_{3,i} \right) =$$

$$= \frac{1}{3} \left(P^{(2)}_{1,1} + P^{(2)}_{2,1} + P^{(2)}_{3,1} \right) + \frac{2}{3} \left(P^{(2)}_{1,2} + P^{(2)}_{2,2} + P^{(2)}_{3,2} \right) + \frac{3}{3} \left(P^{(2)}_{1,3} + P^{(2)}_{2,3} + P^{(2)}_{3,3} \right) +$$

$$+ \frac{4}{3} \left(P^{(2)}_{1,4} + P^{(2)}_{2,4} + P^{(2)}_{1,4} \right) =$$

$$= \frac{1}{3} \left(\frac{5}{72} + 0 + 0 \right) + \frac{2}{3} \left(0 + \frac{7}{72} + \frac{17}{24} \right) + \left(0 + \frac{5}{72} + \frac{7}{24} \right) + \frac{4}{3} \left(\frac{17}{24} + 0 + 0 \right)$$

$$= \frac{5}{36} + \frac{31}{36} + \frac{77}{24} + \frac{17}{18} = \frac{767}{72}$$

3) $\lim_{n \to \infty} P^{(2n)}_{1,3} = 0$

$\lim_{n \to \infty} P^{(2n)}_{1,4} = \pi_4$

Troviamo π_4 sfruttando la matrice P^2:

Siccome lo stato 4 appartiene alla classe $[1] = \{1, 4\}$

Consideriamo la sottomatrice di elementi (di P^2)

$$\begin{pmatrix} (1,1) & (1,4) \\ (4,1) & (4,4) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{5}{72} & \frac{7}{72} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \\ \frac{13}{24} & \frac{11}{24} \end{pmatrix}$$

$$\Rightarrow \begin{cases} \pi_1 = \frac{5}{12}\pi_1 + \frac{13}{24}\pi_4 \\ \pi_4 = \frac{7}{12}\pi_1 + \frac{11}{24}\pi_4 \\ \pi_1 + \pi_4 = 1 \end{cases} \Rightarrow \begin{cases} \frac{7}{12}\pi_1 = \frac{13}{24}\pi_4 \Rightarrow \pi_1 = \frac{13}{14}\pi_4 \\ \pi_4 = \pi_4 \Rightarrow OK! \\ \frac{13}{14}\pi_4 + \pi_4 = 1 \end{cases}$$

$$\Rightarrow \frac{27}{14}\pi_4 = 1 \Rightarrow \pi_4 = \frac{14}{27}$$

Dunque
$$\lim_{n\to\infty} P_{1,4}^{(2n)} = \pi_4 = \frac{14}{27}$$

- $\lim_{n\to\infty} P_{2,3}^{(n)}$

$\Rightarrow$

Se n è PARI : $n = 2K$

$$\Rightarrow P_{2,3}^{(2K)} = \pi_3$$

Se n è DISPARI : $n = 2K+1$

$$\Rightarrow P_{2,3}^{(2K+1)} = \sum_{i=1}^{4} P_{2,i}^{(2K)} P_{i,3}^{(1)} =$$

$$= P_{2,1}^{(2K)} P_{1,3}^{(1)} + P_{2,2}^{(2K)} P_{2,3}^{(1)} + P_{2,3}^{(2K)} P_{3,3}^{(1)} + P_{2,4}^{(2K)} P_{4,3}^{(1)} =$$

$$= 0 + \pi_2 \cdot 0 + \pi_3 \cdot 0 + 0 = 0$$

- $\lim_{n\to\infty} P(X_n = 2)$:

$$P(X_n=2) = \sum_{i=1}^{4} P(X_n = 2 \mid X_0 = i)\, P(X_0 = i) =$$

$$= \frac{1}{3}\Big(P(X_n=2 \mid X_0=1) + P(X_n=2 \mid X_0=2) + P(X_n=2 \mid X_0=3) \Big) =$$

$$= \frac{1}{3}\left(P_{1,2}^{(n)} + P_{2,2}^{(n)} + P_{3,2}^{(n)} \right)$$

Se n è PARI : $n = 2K$

$$\Rightarrow P_{1,2}^{(2K)} = 0$$

$$P_{2,2}^{(2K)} = \pi_2$$

$$P_{3,2}^{(2K)} = \pi_2$$

$$\Rightarrow P(X_n=2) \longrightarrow \frac{1}{3}\left(0 + \pi_2 + \pi_2\right) = \frac{2}{3}\pi_2$$

Se n è **DISPARI**: $n = 2k+1$

$$\Rightarrow p_{1,2}^{(2k+1)} = \sum_{i=1}^{4} p_{1,i}^{(2k)} p_{i,2}^{(1)}$$

$$p_{2,2}^{(2k+1)} = 0$$

$$p_{3,2}^{(2k+1)} = 0$$

$$\Rightarrow p_{1,2}^{(2k+1)} = \sum_{i=1}^{4} p_{1,i}^{(2k)} p_{i,2}^{(1)} =$$

$$= p_{1,1}^{(2k)} p_{1,2}^{(1)} + p_{1,2}^{(2k)} p_{2,2}^{(1)} + p_{1,3}^{(2k)} p_{3,2}^{(1)} + p_{1,4}^{(2k)} p_{4,2}^{(1)} =$$

$$= \pi_1 \frac{1}{2} + 0 + 0 + \pi_4 \frac{3}{4} =$$

$$= \frac{1}{2}\pi_1 + \frac{3}{4}\pi_4 = \frac{1}{2}\frac{13}{14}\pi_4 + \frac{3}{4}\pi_4 =$$

$$= \frac{17}{14}\pi_4$$

$$\Rightarrow P(X_n = 2) \longrightarrow \frac{17}{14}\pi_4$$

▨

Es 13.4

Una catena di Markov $(X_n)_{n \in \mathbb{N}}$ con insieme degli stati $S = \{1,2,3,4,5\}$ ha la seguente matrice di transizione

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 3 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 4 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 5 & \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 \end{array}$$

e distribuzione iniziale

$\mu(1) = 0$
$\mu(2) = \frac{2}{3}$
$\mu(3) = \frac{1}{3}$
$\mu(4) = 0$
$\mu(5) = 0$

a) Dire quali sono le classi di equivalenza e i periodi

b) Calcolare: $\lim_{n\to\infty} p_{1,5}^{(n)}$

$\lim_{n\to\infty} p_{3,5}^{(n)}$

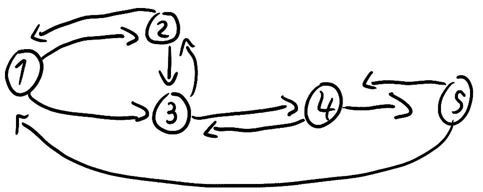
$\lim_{n\to\infty} \left(p_{2,1}^{(n)} + p_{3,5}^{(n)}\right)$

$\lim_{n\to\infty} P(X_n = 5)$

c) Calcolare $P(X_1 \le 2)$ e $P(X_2 = 5)$

__sol__

a)



Tutti gli stati comunicano tra loro

$\Rightarrow$ $\exists !$ classe di equivalenza

Periodo $d = 1$

b) $\lim_{n \to \infty} P_{1,5}^{(n)} = \pi_5$

$\lim_{n \to \infty} P_{3,5}^{(n)} = \pi_5$

$\lim_{n \to \infty} \left(P_{2,3}^{(n)} + P_{3,5}^{(n)} \right) = \pi_3 + \pi_5$

Calcoliamo $\pi_1, \pi_2, \pi_3, \pi_4, \pi_5$

$$\begin{cases} \pi_1 = \frac{1}{2}\pi_2 + \frac{2}{3}\pi_5 \\ \pi_2 = \frac{1}{2}\pi_1 + \frac{2}{3}\pi_3 \\ \pi_3 = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 + \frac{2}{3}\pi_4 \\ \pi_4 = \frac{1}{3}\pi_3 + \frac{1}{3}\pi_5 \\ \pi_5 = \frac{1}{3}\pi_4 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1 \end{cases}$$

$\Rightarrow$

$$\begin{cases} \pi_1 = \frac{1}{2}\pi_2 + \frac{2}{3}\pi_5 \Rightarrow \pi_1 = \frac{1}{2}\left(\frac{4}{3}\pi_5 + \frac{8}{9}\pi_3\right) = \frac{2}{3}\pi_5 + \frac{4}{9}\pi_3 + \frac{2}{3}\pi_5 = \frac{8}{9}\pi_5 + \frac{4}{9}\pi_3 \\ \pi_2 = \frac{1}{2}\left(\frac{1}{2}\pi_2 + \frac{2}{3}\pi_5\right) + \frac{2}{3}\pi_3 = \frac{1}{4}\pi_2 + \frac{1}{3}\pi_5 + \frac{2}{3}\pi_3 \Rightarrow \frac{3}{4}\pi_2 = \frac{1}{3}\pi_5 + \frac{2}{3}\pi_3 \Rightarrow \pi_2 = \frac{4}{9}\pi_5 + \frac{8}{9}\pi_3 \\ \pi_3 = \frac{1}{2}\left(\frac{8}{3}\pi_5 + \frac{4}{9}\pi_3\right) + \frac{1}{2}\left(\frac{4}{3}\pi_5 + \frac{8}{9}\pi_3\right) + \frac{2}{3}\pi_4 = \frac{2}{3}\pi_4 + \frac{4}{9}\pi_5 + \frac{2}{9}\pi_3 + \frac{2}{9}\pi_5 + \frac{4}{9}\pi_3 = \frac{2}{3}\pi_4 + \frac{2}{3}\pi_5 + \frac{2}{3}\pi_3 \Rightarrow \frac{1}{3}\pi_3 = \frac{2}{3}\pi_5 + \frac{2}{3}\pi_4 \Rightarrow \pi_3 = 2\pi_5 + 2\pi_4 \\ \pi_4 = \frac{1}{3}\pi_5 + \frac{1}{3}\pi_3 = \frac{1}{3}\pi_5 + \frac{1}{3}(2\pi_5 + 2\pi_4) = \frac{1}{3}\pi_5 + \frac{2}{3}\pi_5 + \frac{2}{3}\pi_4 \Rightarrow \frac{1}{3}\pi_4 = \pi_5 \Rightarrow \pi_4 = 3\pi_5 \\ \pi_5 = \frac{1}{3} \cdot 3\pi_5 = \pi_5 \Rightarrow OK \end{cases}$$

$\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$

$\Rightarrow$

$\frac{8}{9}\pi_5 + \frac{4}{9}\left(2\pi_5 + 2 \cdot 3\pi_5\right) + \frac{4}{9}\pi_5 + \frac{8}{9}\left(2\pi_5 + 2 \cdot 3\pi_5\right) + 2\pi_5 + 2 \cdot 3\pi_5 + 3\pi_5 + \pi_5 = 1$

$\Rightarrow$

$\frac{8}{9}\pi_5 + \frac{8}{9}\pi_5 + \frac{24}{9}\pi_5 + \frac{4}{9}\pi_5 + \frac{16}{9}\pi_5 + \frac{48}{9}\pi_5 + 2\pi_5 + 6\pi_5 + 3\pi_5 + \pi_5 = 1$

$\Rightarrow 24\pi_5 = 1 \Rightarrow \pi_5 = \frac{1}{24}$

$\pi_3 = 2\pi_5 + 2\pi_4 = 2 \cdot \frac{1}{24} + 2 \cdot 3 \cdot \frac{1}{24} = \frac{1}{3}$

• $\lim_{n \to \infty} P(X_n = 5)$

$P(X_n = 5) = \sum_{i=1}^{5} P(X_n = 5 \mid X_0 = i) \cdot P(X_0 = i) =$

$$= \frac{2}{3} P_{2,5}^{(n)} + \frac{1}{3} P_{3,5}^{(n)} =$$

$$\Rightarrow \lim_{n \to \infty} P(X_n = 5) = \frac{2}{3} \pi_5 + \frac{1}{3} \pi_5 = \pi_5$$

c) $P(X_1 \leq 2) = P(X_1 = 1) + P(X_1 = 2)$

- $P(X_1 = 1) = \sum_{i=1}^{5} P(X_1 = 1 \mid X_0 = i)\, P(X_0 = i) =$

$$= \frac{2}{3} P_{2,1}^{(1)} + \frac{1}{3} P_{3,1}^{(1)} = \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 0 = \frac{1}{3}$$

- $P(X_1 = 2) = \sum_{i=1}^{5} P(X_1 = 2 \mid X_0 = i)\, P(X_0 = i) =$

$$= \frac{2}{3} P_{2,2}^{(1)} + \frac{1}{3} P_{3,2}^{(1)} = 0 + \frac{1}{3} \frac{2}{3} = \frac{2}{9}$$

$$\Rightarrow P(X_1 \leq 2) = \frac{1}{3} + \frac{2}{9} = \frac{5}{9}$$

$$P(X_2 = 5) = \sum_{i=1}^{5} P(X_2 = 5 \mid X_0 = i)\, P(X_0 = i) =$$

$$= \frac{2}{3} P_{2,5}^{(2)} + \frac{1}{3} P_{3,5}^{(2)}$$

$$P^2 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 \end{pmatrix} = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet & \frac{1}{9} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

$$\Rightarrow P(X_2 = 5) = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot \frac{1}{9} = \frac{1}{27}$$

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